The moduli space $\mathcal{M}_n(\Sigma)$ of stable fiber bundles over a compact Riemann surface

A.Balan

Ecole Polytechnique

Centre de mathematiques

UMR 7640 of CNRS

F-91128 PALAISEAU Cedex

email: balan@math.polytechnique.fr

Abstract

The moduli space of holomorphic fiber bundles $\mathcal{M}_n(\Sigma)$ over a compact Riemann surface Σ is considered. A formula for the regularised determinant and an other for the symplectic form at trivial bundle are proposed.

1   Introduction

The moduli space $\mathcal{M}_n(\Sigma)$ of holomorphic stable fiber bundles of rank n and trivial determinant over a compact Riemann surface and of genus $g \geq 2$ possesses usually three descriptions. The first one is given by the *symplectic quotient* [9] of the space of the unitary connections $\mathcal{A}_\Sigma$ over the trivial fiber bundle when the moment action given by the curvature is considered. The second one is a double algebraic quotient called *field* [18] which represents the holomorphic fiber bundles. The third one is furnished by a *quotient of the set of irreductible homomorphisms* [16] of the Poincaré group of the Riemann surface $\pi_1(\Sigma)$ in the group of unitary symmetries of rank n, over the action of the same group. This space is a non-compact complex variety. The regularised determinant gives the corresponding metric, called Quillen metric, a formula for the regularised determinant is proposed in the formula 1. It exits, morover, a symplectic form over this space which is a kaehlerian variety [10]. Here is presented also a formula for the symplectic form at trivial fiber bundle in theorem 5.1 by mean of a reduction of the form by the harmonic ones over the Riemann surface.

2   The cocycles and the operators $\overline{\partial}$

2.1   The connection with a 1-cocycle, $f(z)$

Let a compact Riemann surface Σ be, with a point p and a holomorphic injection of the unitary disc of the complex plan of the surface:

$$i : \quad (D,0) \hookrightarrow (\Sigma, p).$$

Let ϱ a partition of the unity be defined for the disc and the surface minus a point. Let a holomorphic 1-cocycle be with valus in the Lie algebra, $Lie(Sl_n)(\mathbb{C})$ and defined over the disc minus a point:

$$f(z) : [D - \{0\}] \to Lie(Sl_n)(\mathbb{C}).$$

The holomorphic cocycle can then be considered with values in the Lie group, $Sl_n(\mathbb{C})$ [4]:

$$exp(f)(z) : [D - \{0\}] \to Sl_n(\mathbb{C}).$$

Now, a differential operator is defined associated with these datas, acting over the trivial fiber bundle of rank n and the solutions define a holomoprhic fiber bundle equivalent with the fiber bundle which is obtained with a holomorphic cocycle. The following operator is considered:

$$\overline{\partial} + f(z)\overline{\partial}\varrho.$$

This operator is well defined, the partition of unity being constant near the point p and outside the disc D. The connection is defined taking the adjoint for the trivial metric:

$$d + f(z)\overline{\partial}\varrho - f(z)^*\partial\varrho.$$

2.2   The isomorphism $\mathcal{C}^\infty$ with the trivial fiber bundle

The holomorphic fiber bundle is associated with a 1-cocycle:

$$([\Sigma - D].\mathbb{C}^n)\coprod(D.\mathbb{C}^n)/R,$$

where R is the equivalence relation given by the holomorphic cocycle:

$$\forall z \in S^1; \forall s \in \mathbb{C}^n;$$

$$(z,s)R(z,[\exp(f)(z)]s).$$

There is an injection of a section $\mathcal{C}^\infty$ of the fiber bundle in a section of the trivial fiber bundle by mean of the following applications, which define the injection:

$$f_1: \ (z,s) \in ([\Sigma - D].\mathbb{C}^n) \mapsto (z,(\exp(-\varrho f(z)))s).$$

$$f_2: \ (z,s) \in (D.\mathbb{C}^n) \mapsto (z,(\exp(-(1-\varrho)f(z)))s).$$

$$
\begin{array}{ccc}
([\Sigma - D].\mathbb{C}^n)\coprod(D.\mathbb{C}^n)/R & \rightarrow & (\Sigma - D).\mathbb{C}^n \\
\downarrow & & \downarrow f_1 \\
D.\mathbb{C}^n & \xrightarrow{f_2} & \Sigma.\mathbb{C}^n
\end{array}
$$

$$([\Sigma - D].\mathbb{C}^n)\coprod(D.\mathbb{C}^n)/R \rightarrow \Sigma.\mathbb{C}^n,$$

by the universal property of the product in the category of the fiber bundles $\mathcal{C}^\infty$ over the Riemann surface Σ. This arrow is a trivialisation $\mathcal{C}^\infty$ of the fiber bundle. The holomorphic local sections of the

fiber bundle defined with the cocycle are taken by the above defined application in these which verify over Σ^*:

$$[\exp(-\varrho f(z)) \circ \overline{\partial} \circ \exp(\varrho f(z))]\ s = \overline{\partial}s + f(z)\overline{\partial}\varrho\ s = 0.$$

So, a connection is obtained canonicaly associated with the 1-cocycle, taking the adjoint by the metric.

2.3   The distributions

The differential operator which is obtained is dependant of the choice of a partition of the unity. To avoid the choice, the partition of unity must go toward the caracteristic function of the disc, in the sens of the distributions [17], so an operator of the type is obtained:

$$\overline{\partial} + [f(z)\delta_{S^1}/2\pi]d\overline{z}:$$

$$\Gamma^0(\Sigma, \mathbb{C}^n) \longrightarrow \Gamma^{0,1}(\Sigma, \mathbb{C}^n)\hat{\otimes}_{\mathcal{C}^\infty(\Sigma)}\mathcal{D}'(\Sigma, \mathbb{C}).$$

Where $\mathcal{D}'(\Sigma, \mathbb{C})$ is the space of distributions and $\delta_{S^1}/2\pi$, is a distribution associated with the circle. The tensor product is taken with the ring of functions $\mathcal{C}^\infty$ over Σ. So the operator of derivation plus an operator with values in the distributions over the Riemann surface Σ is obtained. The whole connection is obtained taking the adjoint of

the operator for the metric over the trivial fiber bundle; it defines the part of type $(1,0)$ with compatibility with the condition of unitarity of the connection:

$$d + f(z)\overline{\partial}\varrho - f(z)^*\partial\varrho.$$

To avoid the choice of a partition of unity, it must go towards the caracteristic function of the disc; then an operator with values in the distributions of the following type is obtained:

$$d + [f(z)\delta_{S^1}d\overline{z} - f(z)^*\delta_{S^1}dz]/2\pi:$$

$$\Gamma^0(\Sigma, \mathbb{C}^n) \longrightarrow \Gamma^1(\Sigma, \mathbb{C}^n)\hat{\otimes}_{\mathcal{C}^\infty(\Sigma)}\mathcal{D}'(\Sigma, \mathbb{C}).$$

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3 The tangent space, $T\mathcal{M}_n(\Sigma)$

With the different descriptions of the space of moduli of the fiber bundles of the Riemann surface Σ, the tangent space of the moduli, $T\mathcal{M}_n(\Sigma)$, can be presented.

In the first case, some complex forms of type $(0, 1)$ over the Riemann surface Σ, taken as elements of the tangent space to the connections space, $\mathcal{A}_\Sigma$.

In the second case, the Lie algebra of the groups of infinite dimension which is considered $Lie(Sl_n)((z))$, and the injections.

In the third case:

$$Hom(\pi_1(\Sigma), Lie(SU_n)(\mathbb{C})).$$

4 The Quillen metric

The Quillen metric is a metric over the determinant fiber bundle of the moduli space, and is given by regularised determinants. Let a Riemann surface Σ be, with boundary S^1. The derivation of the regularised determinant is calculated [1]for a family of connections for a trivial holomorphic structure and a boundary condition, for example of Dirichlet type.

4.1 The regularised determinants

The regularised determinant of a infinite positiv autoadjoint operator P over a hibert space of infinite dimension is defined by the proper values λ_i, $i \in \mathbb{N}^*$. First the theta function of the operator is defined:

$$\theta_P(t) = \sum_{i \in \mathbb{N}^*} \exp(-t\lambda_i) \ , \ t \in \mathbb{R}_+^*.$$

Where the zero proper values are taken off. This series of positiv terms is supposed convergent for all t. The theta function is supposed to admit an asymptotic development when t goes towards zero in the following form:

$$\theta_P(t) = \sum_n a_n t^n.$$

Where n takes its values in the arithmetic increasing sequence of rationals. The zeta function of the operator ζ_P, which is the Mellin transform of the theta function, is then well defined and has an analytic prologation over the complex plan with poles in the numbers $-n + 1$. In the case where zero is a regular value, the regularised determinant is defined as:

$$det'(P) = \exp(-\zeta'(0)).$$

The definition can be applied in the case of a Laplacian with a connection over the above fiber bundle of a Riemann surface. The theta function is well defined from the fact that the heat kernel gives a trace operator. The asymptotic development is the one of Minakshisundaram-Pleijel.

4.2 The case of rank 1

First the case of a one dimensional fiber bundle is considered. The family of connections over the trivial fiber bundle is of the form:

$$A_t^{0,1} = \bar{\partial} + t\bar{\partial}f = \exp(-tf) \circ \bar{\partial} \circ \exp(tf).$$

Where f is a real function over the Riemann surface. The part of type $(1,0)$ being given by the adjoint by mean of the canonical metric over the trivial fiber bundle. The considered Laplacian are then:

$$\Delta_t = \exp(tf) \circ \bar{\partial}^* \circ \exp(-2tf) \circ \bar{\partial} \circ \exp(tf).$$

A Dirichlet condition is imposed over the boundary to obtain a positiv autoadjoint operator. The regularised determinant of the family of operators is then derivable. An expression for the derivation is seeked. The following formula is taken:

$$\int_\epsilon^\infty \theta(t)\frac{dt}{t} = \int_0^\infty \theta(t)\frac{dt}{t} + \sum_{n\leq 0}(\frac{a_n}{n} - \frac{a_n\epsilon^n}{n}) + a_0 \ log(\epsilon) + \int_\epsilon^1 r(t)\frac{dt}{t}.$$

Where r is the theta function without divergent terms in zero. The following equality is deduced:

$$\int_\epsilon^\infty \theta(t)\frac{dt}{t} = \zeta'(0) + \sum_{n\leq 0} a_n\frac{\epsilon^n}{n} + a_0 log(\epsilon) + \int_0^\epsilon r(t)\frac{dt}{t}.$$

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Then, a derivation with respect to the the parameter of the family of operators, s:

$$\int_\epsilon^\infty \frac{d}{ds}\theta(t)\frac{dt}{t} = \frac{d}{ds}\zeta'(0) + \sum_{n\leq 0}\frac{d}{ds}a_n\frac{\epsilon^n}{n} + \frac{d}{ds}a_0 log(\epsilon) + \int_0^\epsilon \frac{d}{ds}r(t)\frac{dt}{t}.$$

So:

$$\frac{d}{ds}\zeta'(0) = PF\int_\epsilon^\infty \frac{d}{ds}\theta(t)\frac{dt}{t}.$$

Where PF is the finite part when ϵ near zero. $\theta(t) = tr(\exp(-t\Delta))$, where the heat kernel associated with the Laplacian is considered, it is finaly obtained:

$$\frac{d}{ds}\zeta'(0) = PF tr((\frac{d}{ds}\Delta)\Delta^{-1}\exp(-\epsilon\Delta)).$$

Let now $|\Phi_i\rangle, i \in \mathbb{N}^*$, an orthogonal basis of proper vectors of the positiv autoadjoint operator Δ for the proper values λ_i; for this basis, the trace is expressed as:

$$tr((\frac{d}{ds}\Delta)\Delta^{-1}\exp(-\epsilon\Delta) = \sum_i \langle\Phi_i|(\frac{d}{ds}\Delta)\Delta^{-1}\exp(-\epsilon\Delta)|\Phi_i\rangle =$$

$$= \sum_i \langle\Phi_i|(\frac{d}{ds}\Delta)|\Phi_i\rangle\lambda_i^{-1}\exp(-\epsilon\lambda_i).$$

As the $|\Phi_i\rangle$ are proper vectors of the operator Δ, it holds:

$$\Delta|\Phi_i\rangle = \lambda_i|\Phi_i\rangle.$$

Derivating, it is obtained that:

$$(\frac{d}{ds}\Delta)|\Phi_i\rangle + \Delta(\frac{d}{ds}|\Phi_i\rangle) = (\frac{d}{ds}\lambda_i)|\Phi_i\rangle + \lambda_i(\frac{d}{ds}|\Phi_i\rangle).$$

The variations of the operators Δ are then in the form:

$$\frac{d}{ds}\Delta = f\exp(sf)\circ\bar{\partial}^*\circ\exp(-2sf)\circ\partial\circ\exp(sf)+$$

$$+\exp(sf)\circ\bar{\partial}^*\circ(-2f)\exp(-2sf)\circ\bar{\partial}\circ\exp(sf)+$$

$$+\exp(sf)\circ\bar{\partial}^*\circ\exp(-2sf)\circ\bar{\partial}\circ\exp(sf)f.$$

This expression is put in the calculation of the trace and using the fact that the vectors $|\Phi_i\rangle$ are unitary, it is obtained:

$$PFtr((\frac{d}{ds}\Delta)\Delta^{-1}\exp(-\epsilon\Delta)).$$

Let now $|\Phi_i\rangle, i \in \mathbb{N}^*$, an orthogonal basis of proper vectors of the positiv autoadjoint operator Δ for the proper values λ_i; for this basis, the trace is expressed as:

$$tr((\frac{d}{ds}\Delta)\Delta^{-1}\exp(-\epsilon\Delta) = \sum_i\langle\Phi_i|(\frac{d}{ds}\Delta)\Delta^{-1}\exp(-\epsilon\Delta)|\Phi_i\rangle =$$

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$$(\frac{d}{ds}\Delta)|\Phi_i\rangle + \Delta(\frac{d}{ds}|\Phi_i\rangle) = (\frac{d}{ds}\lambda_i)|\Phi_i\rangle + \lambda_i(\frac{d}{ds}|\Phi_i\rangle).$$

The variations of the operators Δ are then in the form:

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$$+\exp(sf)\circ\bar{\partial}^*\circ(-2f)\exp(-2sf)\circ\bar{\partial}\circ\exp(sf)+$$

$$+\exp(sf)\circ\bar{\partial}^*\circ\exp(-2sf)\circ\bar{\partial}\circ\exp(sf)f.$$

This expression is put in the calculation of the trace and using the fact that the vectors $|\Phi_i\rangle$ are unitary, it is obtained:

$$\sum_i\langle\Phi_i|(\frac{d}{ds}\Delta)|\Phi_i\rangle\lambda_i^{-1}\exp(-\epsilon\lambda_i) =$$

$$= 2\sum_i\langle\Phi_i|f|\Phi_i\rangle\exp(-\epsilon\lambda_i) - 2\sum_i\langle\Psi_i|f|\Psi_i\rangle\exp(-\epsilon\lambda_i).$$

Where $|\Psi_i\rangle = \exp(-sf)\circ\partial\circ\exp(sf)|\Phi_i\rangle/\sqrt{\lambda_i}$ are the orthogonal vectors and the propeer vectors for the proper values λ_i of the operator:

$$\Delta^- = \exp(-sf)\circ\bar{\partial}\circ\exp(2sf)\circ\bar{\partial}^*\circ\exp(-sf),$$

acting over the 1-forms over the surface. The variations of the determinant are then:

Formula 4.1 .

$$\frac{d}{ds}\zeta'(0) = PF\ 2tr(f\exp(-\epsilon\Delta)) - PF\ 2tr(f\exp(-\epsilon\Delta^-)). \quad (1)$$

To obtain an explicite expression of the derivation of the regularised determinant, the first coefficients of the asymptotic development of the heat kernel must be calculated when the parameter goes toward zero by mean of the Seeley methods with boundary terms.

4.3 The case of higher ranks

Let the trivial fiber bundle be of rank n and a connection put in the following form:

$$A^{0,1} = \bar{\partial} + \alpha^{-1}\bar{\partial}\alpha.$$

Where α is an inversible matrix. A polar decomposition is applied for the matrix:

$$\alpha = H \circ U.$$

Where U is unitary and the H hermitian. The connection is then put in the following form:

$$U^* \circ H^{-1} \circ \bar{\partial} \circ H \circ U.$$

And the associated Laplacian is:

$$U^* \circ H \circ \bar{\partial}^* \circ H^{-2} \circ \bar{\partial} \circ H \circ U.$$

For the calculation of the regularised determmiant, a positiv hermitian matrix is taken and the parametrised family of Laplacian is considered with the real t:

$$H^t \circ \bar{\partial}^* \circ H^{-2t} \circ \bar{\partial} \circ H^t.$$

So a similary calculation is done.

An expression for the derivation of the regularised determinant has been obtained in the case of a conjugated connection over a Riemann surface with boundary. The derivation could be integrated [5] for an exatc formula of the regularised determinant.

5 The symplectic form of the moduli space $\mathcal{M}_n(\Sigma)$

5.1 The symplectic form of the space of moduli, ω

When the holomorhic fiber bundles are considered (topologicaly trivial) as being connections over the trivial fiber bundle, the symplectic form can be considered as being given by the natural symplectic form of the space of unitary connections:

$$\omega(\alpha_1, \alpha_2) = \int_\Sigma tr(\alpha_1 \wedge \alpha_2).$$

Where α_1 and α_2 are some 1-forms with values in the anti-symmetric endomorphisms of the trivial fiber bundle considered as elements of the tangent space of the space of connections. The space of moduli of the fiber bundles $\mathcal{M}_n(\Sigma)$ can be considered as a symplectic quotient of the space of connections and the tangent space is then identified with the harmonic 1-forms with values in the antisymmetric endomorphisms of the fiber bundle:

$$H^1(\Sigma, A(\mathbb{C}^n)).$$

5.2 The reduction of a holomorphic 1-cocycle in a 1-form over the surface Σ

There is no actual canonical and explicit mean to associate a connection with a holomorphic fiber bundle (stable). The problem is equivalent to find a metric of the stable holomorphic fiber bundle which allows to show him as an unitary representation of the Poincaré group. But, at the level of the tangent spaces in the trivial fiber bundle, the equivalent of the identification is simple, it is the isomorphism between the cohomology of Čech and the cohomology of de Rham obtained via the Hodge theory [11]. The reduction in a harmonic 1-form is then: let a holomorphic 1-cocycle f be at the point p and a choice of a partition of the unity adapted with the open sets of the surface which are the disc D and the surface Σ minus the point p. There is then a function ϱ with value 1 near the point and zero outside. Then the form of type $(0,1)$ is considered: $f\overline{\partial}\varrho$ reduced in its harmonic

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counterpart in the Dolbeault-Grothendieck cohomology [15]:

$$\phi(Q) = \int_{P \in \Sigma} f(P) \overline{\partial}_P \varrho(P) \wedge \partial_P \overline{\partial}_Q g(P, Q).$$

Where g is the Green function of the Riemann surface Σ. The harmonic form of the de Rahm cohomology is then: $\alpha = Re(\phi)$.

5.3 The reduction of the symplectic form of the double quotient

Let two holomorphic fiber bundles be f_1 and f_2 considered as meromorhic functions over the disc D and a pole in zero.

Definition 5.1 . $: g(P, Q) :$ *is the renormalisation of the Green function de Green:*

$$: g(P, Q) := \ g(P, Q) - ln(|P - Q|),$$

where $P, Q \in \Sigma$ and $|P - Q|$ is the geodesic distance over the Riemann surface Σ.

Lemma 5.1 . *The Green function being biharmonic, it can be written $\partial_z \overline{\partial}_t : g(z, t) :$ as double series holomorphic, anti-holomorphic.*

Proof: the Green function de Green being biharmonic is the solution of the Laplacian $\Delta = \partial \overline{\partial} = \overline{\partial} \partial$. The partial derivations can then be written over the disc as holomorphic function in one of the variables and anti-holomorphic with the other; it shows, by the Cauchy formula, that the expression can be developped near zero in a double series.

$$\partial_z \overline{\partial}_t : g(z, t) := \sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} a_{n,m} z^n \overline{t}^m.$$

The symplectic form is then given by:

Theorem 5.1 . *The symplectic form corresponding with the tangent space of the double quotient at level of the trivial fiber bundle is:*

$$\omega_e(f^1, f^2) = (2\pi)^2 Re(\sum_n \sum_m a_{n,m} tr([f^1_{n-1}]^* f^2_{m-1})), \qquad (2)$$

for $f_1(z) = \sum_{r=q}^{+\infty} f^1_r z^r$ and $f_2(t) = \sum_{l=s}^{+\infty} f^2_l t^l$.

Proof: the symplectic form is the real part of:

$$\int_{Q\in\Sigma} tr((\int_{P\in\Sigma} f^*_1(P)\partial_P\varrho(P) \wedge [\overline{\partial}_P\partial_Q g(P,Q))] \wedge$$

$$\wedge (\int_{P\in\Sigma} f_2(P)\overline{\partial}_P\varrho(P) \wedge [\partial_P\overline{\partial}_Q g(P,Q)])).$$

There is a function ϱ with 1 value near the point and zero outside of the disc, the function goes towards the caracteristic function of the disc. Stockes is aplied two times and the property of the function of Green is used [15]:

$$\int_{Q\in\Sigma} [\partial_P\overline{\partial}_Q g(P,Q)] \wedge [\partial_Q\overline{\partial}_{Q'} g(Q,Q')] = \partial_P\overline{\partial}_{Q'} g(P,Q').$$

Instead an integration over all the surface Σ, minus where ϱ is 1, as ϱ goes towards the caracteristic function of the disc, Stockes is applied to $[\Sigma - D]$. The following expression of the symplectic form is obtained:

$$\int_{P\in S^1} \int_{Q\in S^1} tr(f^*_1(P)f_2(Q))[\partial_P\overline{\partial}_Q : g(P,Q) :].$$

$\partial_P\overline{\partial}_Q : g(P,Q) :$ is supposed being developped in a double series over the disc or a smaller; this double integrale is explained around zero:

$$\int\int_{z,t\in S^1} tr(f^*_1(z)f_2(t)) \sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} a_{n,m} z^n \overline{t}^m dz d\overline{t}.$$

And so,

$$\sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} \sum_{r=q}^{+\infty} \sum_{l=s}^{+\infty} a_{n,m} tr([f^1_r]^* f^2_l) \int\int_{z,t\in S^1} z^n \overline{t}^m \overline{z}^r t^l dz d\overline{t},$$

developping in series the functions f_1 and f_2. So:

$$(2\pi)^2 \sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} \sum_{r=q}^{+\infty} \sum_{l=s}^{+\infty} a_{n,m} tr([f^1_r]^* f^2_l)\delta_{n-1,r}\delta_{m-1,l},$$

with δ, the symbol of Kronecker.

6   Conclusion

A presentation of the moduli spaces, $\mathcal{M}_n(\Sigma)$, over a Riemann surface Σ has been done, and it has been showed how to obtain them in three different ways. A formula for the derivative of the regularised determinant has been computed. The existence of a simple formula has been showed for the symplectic structure at level of the tangent space at the trivial fiber bundle of the double quotient.

7   Acknowledgements

A   The prodistributions

A.1   The product of two distributions

Let a compact differentiable variety M be, with orientation, the dual $\mathcal{D}'(M)$ [17] of the functions $\mathcal{C}^\infty$ is considered with compact support.

Definition A.1 . $\mathbb{D}'_2(M)$, *is the space of* 2-*distributions, the following topological tensor product* [7]:

$$\mathbb{D}'_2(M) := \mathcal{D}'(M)\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathcal{D}'(M).$$

This space, which is a module over the ring of functions $\mathcal{C}^\infty$, allows a definition of a product of two distributions in the following sens:

The product of two distributions η, η' is the element $\eta\eta'$:

$$\eta\eta' = \eta.\eta' = \eta \otimes \eta' \in \mathbb{D}'_2(M).$$

A.2   The product of n distributions

Definition A.2 . $\mathbb{D}'_n(M)$ *is the space of n-distributions, the following tensor product:*

$$\mathbb{D}'_n(M) := \mathcal{D}'(M)\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathcal{D}'(M)\hat{\otimes}_{\mathcal{C}^\infty(M)}...\hat{\otimes}_{\mathcal{C}^\infty(M)}\mathcal{D}'(M).$$

e over the ring of functions $\mathcal{C}^\infty$, allows a definition of a product of two distributions in the following sens:

The product of two distributions η, η' is the element $\eta\eta'$:

$$\eta\eta' = \eta.\eta' = \eta \otimes \eta' \in \mathbb{D}'_2(M).$$

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The tensor product is taken n times. This space, a module over the ring of functions $\mathcal{C}^\infty$, allows a definition of the product of n distributions in the following sens:

The product of n distributions η_1, η_2, ..., η_n is the element $\eta_1\eta_2...\eta_n$:

$$\eta_1\eta_2...\eta_n = \eta_1.\eta_2.\ ...\ .\eta_n = \eta_1 \otimes \eta_2 \otimes ... \otimes \eta_n \in \mathbb{D}'_n(M).$$

A.4   The space $\mathbb{D}'(M)$ of prodistributions

$$\mathbb{D}'_n(M).\mathbb{D}'_m(M) \to \mathbb{D}'_{n+m}(M).$$

Definition A.4 . $\mathbb{D}'(M)$, *module over* $\mathcal{C}^\infty(M)$, *space of prodistributions is the following one:*

$$\mathbb{D}'(M) := lim_{n\in\mathbb{N}, n\geq 2}\ \mathbb{D}'_n(M).$$

The limit is taken with the system of the inclusions [7].

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